

# Balanced Realization of Lossless Systems

## Schur Parameters, Canonical Forms and Applications

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# INTRODUCTION

# Motivation

Why are we interested in balanced realizations of **lossless systems**?

Because lossless systems play an important role in the construction of well behaved **parameterizations** and **state-space canonical forms** for various classes of **stable** linear systems.

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## Further motivation

A **well behaved** parameterization for a class of stable linear systems should exhibit nice properties with respect to:

- **Model order selection, model order reduction**
  - recursive construction to obtain higher orders
  - reduced order models again in canonical form
  - well aligned with (balance-and-)truncate type procedures
- **System identification and numerical optimization** of a criterion
  - straightforward computation of derivatives
  - good numerical conditioning
  - no singularities, well understood limit behavior
- **Structure** built into the model class
  - stability
  - Kronecker structure
  - prescribed pole locations

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# Application areas

Lossless systems show up in research areas and application domains such as:

- Interpolation theory
- System identification: separable least squares
- L2-approximation
- Orthogonal wavelet theory

The theory of lossless systems has a very rich mathematical structure!

## Some history

- A **balanced** canonical form for **scalar** lossless functions in **continuous-time** was first presented by **Ober (1987)**.
- **Hanzon and Ober (1997, 1998)** constructed parameterizations for various other classes of systems in continuous-time. Here, the realizations had a **positive upper triangular reachability matrix**. In the **MIMO case**, balanced overlapping canonical forms were constructed using Kronecker indices and nice selections.
- The **discrete-time** case can be obtained from the continuous-time case by a bilinear transformation. However: upper triangularity of the reachability matrix will be lost. And: state vector truncation will no longer return a reduced order system in balanced canonical form.
- In **Hanzon and Peeters (2000)** the **scalar discrete-time** lossless case was handled directly. A balanced canonical form was constructed with a positive upper triangular reachability matrix. It was parameterized with **Schur parameters**.

## Some history (2)

- The class of **inner functions** is bijectively related to the class of lossless functions in a simple way. In another line of research (interpolation theory), [Alpay, Baratchart and Gombani \(1994\)](#) constructed an atlas of overlapping parameter charts for **MIMO inner functions**, using the **tangential Schur algorithm**.
- In [HOP \(2006\)](#) a general framework was developed. The tangential Schur algorithm is used to deal with the **MIMO discrete-time lossless** case, yielding overlapping charts of balanced state-space parameterizations. The interpolation conditions are at points inside the open unit disk.
- In [HOP \(2007\)](#) it is established which choices in this framework yield positive upper triangular reachability matrices, and how **staircase forms** for arbitrary **Kronecker structures** are obtained. In [HOP \(SYSID 2009\)](#) it is shown that more general choices will still produce convenient canonical forms, even when no staircase forms emerge.

## Some history (3)

- In [HOP \(2008\)](#), using the interpolation theoretic results of [Ball, Gohberg and Rodman \(1990\)](#), the framework was generalized to handle interpolation conditions on the stability boundary at infinity. In this way, the original scalar continuous-time balanced canonical form of [Ober \(1987\)](#) is reobtained.
- Recently, these results were generalized to the MIMO case, yielding staircase forms for the reachability matrix in the **continuous-time** lossless case, for arbitrary Kronecker structures.

Alongside the development of theory, it was investigated how these results can be applied in numerical software to handle approximation and identification problems more efficiently. The software RARL2 was published at INRIA; see [Olivi and Marmorat \(2003\)](#) for L2-approximation and model order reduction.

## CANONICAL FORMS AND BALANCING FOR STATE-SPACE SYSTEMS

## Notation for state-space systems

**Discrete-time** LTI state-space system ( $t \in \mathbb{Z}$ ):

$$x_{t+1} = Ax_t + Bu_t,$$

$$y_t = Cx_t + Du_t.$$

**Continuous-time** LTI state-space system ( $t \in \mathbb{R}$ ):

$$x'(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t).$$

State-space systems are denoted by  $(A, B, C, D)$  but also by the **realization matrix**

$$R = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

# Minimality, order, input-output equivalence

- The **transfer function** of a system  $(A, B, C, D)$  is given by

$$G(z) = D + C(zI_n - A)^{-1}B$$

- $(A, B, C, D)$  is **minimal** if no other realization of  $G(z)$  exists for which the state-space dimension  $n$  is smaller.
- This minimal state-space dimension  $n$  is called the **order** of the system. It is equal to the **(McMillan) degree** of the transfer function  $G(z)$ .
- Two systems  $(A, B, C, D)$  and  $(A', B', C', D')$  are **i/o-equivalent** if they yield the same transfer function  $G(z)$ .
- Two minimal systems  $(A, B, C, D)$  and  $(A', B', C', D')$  are i/o-equivalent iff

$$(A', B', C', D') = (TAT^{-1}, TB, CT^{-1}, D)$$

for some invertible state-space transformation matrix  $T$ .

# State-space canonical forms

- A **state-space canonical form** is obtained by specifying a **unique** element  $(A, B, C, D)$  from each i/o-equivalence class; i.e., **for each  $G(z)$** .
- To bring a given  $(A, B, C, D)$  into canonical form, one should be able to compute the transformation matrix  $T$  which does this.
- Conversely, specifying a computational procedure for such  $T$  may serve to construct a canonical form.
- Many state-space canonical forms exist. Especially in the MIMO case, canonical forms usually apply to a generic subset of systems only.
- The choice of a canonical form has impact on further computational procedures and numerical conditioning (e.g.: in system identification, optimization, approximation, or hardware implementation).

# Balancing

A minimal and stable realization  $(A, B, C, D)$  is called **balanced** (Moore (1981); Pernebo and Silverman (1982)) if the controllability and observability Gramians are equal and diagonal:

$$W_c = W_o = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$$

with  $\sigma_1, \sigma_2, \dots, \sigma_n > 0$  the (Hankel) singular values of the system.

In **discrete-time** the (Lyapunov-)Stein equations are given by:

$$\begin{aligned}W_c - AW_cA^* &= BB^*, \\W_o - A^*W_oA &= C^*C.\end{aligned}$$

In **continuous-time** the Lyapunov equations are given by:

$$\begin{aligned}AW_c + W_cA^* &= -BB^*, \\A^*W_o + W_oA &= -C^*C.\end{aligned}$$

## Balancing (2)

- Every minimal system can be brought into balanced form. This form is not entirely unique: sign choices and orderings of the singular values are still possible.
- More importantly: uniqueness is lost when there are singular values with a multiplicity larger than 1. This happens for lossless systems!

## Balancing (3)

**Algorithm** to balance a minimal and stable system  $(A, B, C, D)$ :

- 1 Compute the controllability Gramian  $W_c$ .
- 2 Factor  $W_c$  as  $W_c = P^*P$  (e.g., SVD or Cholesky decomposition).
- 3 Use  $P$  to transform  $(A, B, C, D)$  to get  $W_c = I_n$  (**input normality**).
- 4 Compute the observability Gramian  $W_o$ .
- 5 Diagonalize  $W_o$  with a **unitary** matrix  $Q$  (e.g., SVD or eigensystem computation)  $W_o = Q^*\Sigma^2Q$  with  $\Sigma > 0$  diagonal.
- 6 Use  $Q$  to transform  $(A, B, C, D)$ , to get  $W_c = I_n$  and  $W_o = \Sigma^2$ .
- 7 Use  $\Sigma^{1/2}$  to achieve balancing: to get  $W_c = W_o = \Sigma$ .

# Model order reduction: balance-and-truncate

## Algorithm:

- 1 Bring  $(A, B, C, D)$  into **balanced form** with ordered singular values:  
 $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ .
- 2 Select a **reduced order**  $k < n$ . (E.g., by inspecting and thresholding the singular values  $\sigma_j$ .)
- 3 **Partition** the realization matrix accordingly as:

$$\left( \begin{array}{c|cc} D & C_1 & C_2 \\ \hline B_1 & A_{11} & A_{12} \\ B_2 & A_{21} & A_{22} \end{array} \right)$$

- 4 **Truncate** the last  $n - k$  state vector components to get:

$$\left( \begin{array}{cc} D & C_1 \\ B_1 & A_{11} \end{array} \right)$$

## Model order reduction: balance-and-truncate (2)

- **Key property of the balance-and-truncate procedure:**

In **continuous-time**, the reduced order model is **again balanced** and the  $k$  largest singular values are preserved.

- In **discrete-time** this does not hold: **balancedness is lost**.

- A work around procedure for discrete-time systems:

- 1 Use the bilinear transform to bring the system to continuous-time, preserving balancedness and the Gramians.
- 2 Perform balance-and-truncate in continuous-time.
- 3 Use the inverse bilinear transform to take the reduced order model back to discrete-time.

## Bilinear transformation between discrete-time and continuous-time

- Bilinear transform:  $z \mapsto s = \frac{z-1}{z+1}$ .  
Inverse bilinear transform:  $s \mapsto z = \frac{1+s}{1-s}$ .
- If  $(A, B, C, D)$  realizes the stable discrete-time function  $G(z)$ , then  $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$  realizes the continuous-time stable function  $\hat{G}(s) = G(\frac{1+s}{1-s})$ :

$$\begin{pmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{pmatrix} = \begin{pmatrix} D - C(A + I_n)^{-1}B & \sqrt{2}C(A + I_n)^{-1} \\ \sqrt{2}(A + I_n)^{-1}B & (A - I_n)(A + I_n)^{-1} \end{pmatrix}$$

Conversely:

$$\begin{pmatrix} D & C \\ B & A \end{pmatrix} = \begin{pmatrix} \hat{D} + \hat{C}(I_n - \hat{A})^{-1}\hat{B} & \sqrt{2}\hat{C}(I_n - \hat{A})^{-1} \\ \sqrt{2}(I_n - \hat{A})^{-1}\hat{B} & (I_n + \hat{A})(I_n - \hat{A})^{-1} \end{pmatrix}$$

## LOSSLESS SYSTEMS

## Discrete-time lossless functions

A  $p \times p$  rational matrix function  $G(z)$  is called **lossless** if:

$$G(z)^* G(z) \geq I_p, \quad \text{for } |z| < 1,$$

$$G(z)^* G(z) = I_p, \quad \text{for } |z| = 1,$$

$$G(z)^* G(z) \leq I_p, \quad \text{for } |z| > 1.$$

Some properties:

- The **Bode magnitude plot** is constant:  $|G(e^{i\Omega})| = 1$  for all real  $\Omega$ .  
 $G(z)$  is unitary at each point  $z$  on the unit circle.
- **Stability** holds: all the poles of  $G(z)$  are inside the open unit disk.
- The **inverse** exists:  $G(z)^{-1} = G^*(z^{-1})$ . This is an **inner** function.
- $G(z)$  of degree  $n$  can be factored into a product of  $n$  lossless matrices of degree 1: the **Potapov factorization**.
- $\det(G(z))$  is a scalar lossless function of degree  $n$ .
- The Hankel singular values are all equal to 1.

## Balanced realization of discrete-time lossless functions

### ■ THEOREM

$R = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$  is a balanced minimal realization matrix of a lossless system if and only if:

- (1)  $R$  is unitary,
- (2)  $A$  is asymptotically stable.

Then  $W_c = W_o = I_n$ .

- If  $R$  is unitary and  $A$  has an eigenvalue on the unit circle, then  $G(z)$  is still lossless but of degree  $< n$  (minimality is lost).
- **Unitary state-space transformations** generate all i/o-equivalent balanced realizations of a lossless system  $G(z)$ . This freedom can be used to build additional properties into the realization.
- We use this freedom to make the reachability matrix  $K = (B, AB, A^2B, \dots, A^{n-1}B)$  **positive upper triangular**.

# State vector truncation

## ■ THEOREM

An  $n \times nm$  reachability matrix  $K = (B, AB, A^2B, \dots, A^{n-1}B)$  is positive upper triangular, if and only if the  $n \times (m+n)$  rectangular matrix  $(B, A)$  is positive upper triangular.

- State vector truncation leaves positive upper triangularity of  $(B, A)$  unaffected. Therefore it also returns a positive upper triangular reachability matrix.

## Continuous-time lossless functions

A rational  $p \times p$  matrix function  $G(s)$  is called **lossless** if:

$$G(s)^* G(s) \geq I_p, \quad \text{for } \operatorname{Re}(s) < 0,$$

$$G(s)^* G(s) = I_p, \quad \text{for } \operatorname{Re}(s) = 0,$$

$$G(s)^* G(s) \leq I_p, \quad \text{for } \operatorname{Re}(s) > 0.$$

Some properties:

- The **Bode magnitude plot** is constant:  $|G(i\omega)| = 1$  for all real  $\omega$ .  
 $G(s)$  is unitary at each point  $s$  on the imaginary axis.
- **Stability**: all the poles of  $G(s)$  are in the open left half plane.
- The **inverse** exists:  $G(s)^{-1} = G^*(-s)$ .

## Balanced realization of continuous-time lossless functions

- A balanced realization  $(A, B, C, D)$  of a **continuous-time** lossless function  $G(s)$  is characterized by the following identities:

$$A + A^* = -BB^*,$$

$$C = -DB^*,$$

$$DD^* = I_n.$$

- The two Gramians are again given by  $W_c = W_o = I_n$ .
- **Unitary state-space transformations** generate all i/o-equivalent balanced realizations of a lossless system  $G(s)$ . This freedom can be used to make the reachability matrix  $K$  positive upper triangular.
- **State vector truncation preserves both balancedness and positive upper triangularity of  $K$ .**

## Example: Ober's real scalar canonical form

- Real scalar lossless functions  $G(s)$  of degree  $n$  are of the form:

$$G(s) = -s_1 \frac{s^n - a_1 s^{n-1} + \dots + (-1)^{n-1} a_{n-1} s + (-1)^n a_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

with all the poles of  $G(s)$  in the left half plane and  $s_1 = \pm 1$  is just a sign parameter.

- The associated balanced canonical form of [Ober \(1987\)](#) has the realization matrix  $R$  given by:

$$R = \left( \begin{array}{c|cccccc} -s_1 & s_1 b_1 & 0 & 0 & \dots & \dots & 0 \\ \hline b_1 & -\frac{b_1^2}{2} & \alpha_1 & & & & \\ 0 & -\alpha_1 & 0 & \alpha_2 & & & 0 \\ 0 & & -\alpha_2 & 0 & & & \\ \vdots & & & & \ddots & & \\ \vdots & & & & & & 0 & \alpha_{n-1} \\ 0 & & & & & & -\alpha_{n-1} & 0 \end{array} \right)$$

## Example: Ober's scalar canonical form (2)

This balanced canonical form has the following properties:

- The given parameterization is quite straightforward, involving  $n$  positive real parameters  $b_1, \alpha_1, \dots, \alpha_{n-1}$ .
- The matrix  $A$  is in **tridiagonal Schwarz form**.
- State vector truncation yields a balanced lower order lossless system with similar properties.
- Both  $(B, A)$  and the reachability matrix  $K = (B, AB, A^2B, \dots, A^{n-1}B)$  are upper triangular (with prescribed signs for the entries on their main diagonals).

The latter property allows us to bring any given lossless state-space system into this balanced canonical form with straightforward tools from numerical linear algebra: (1) first make the system input normal, (2) then apply QR-decomposition to  $K$  to make it upper triangular.

## Example: scalar discrete-time lossless systems

Scalar discrete-time lossless functions  $G(z)$  of degree  $n$  are of the form:

$$G(z) = \alpha \frac{\overline{a_n}z^n + \overline{a_{n-1}}z^{n-1} + \dots + \overline{a_1}z + 1}{z^n + a_1z^{n-1} + \dots + a_{n-1}z + a_n}$$

with all the poles of  $G(z)$  in the open unit disk and  $|\alpha| = 1$ .

The associated balanced canonical form for  $G(z)$  of [Hanzon and Peeters \(2000\)](#) has a realization matrix  $R$  explicitly given by:

$$\left( \begin{array}{c|cccccc} \overline{\gamma_n} & \kappa_n \overline{\gamma_{n-1}} & \kappa_n \kappa_{n-1} \overline{\gamma_{n-2}} & \cdots & \cdots & \kappa_n \cdots \kappa_2 \overline{\gamma_1} & \kappa_n \cdots \kappa_1 \overline{\gamma_0} \\ \hline \kappa_n & -\gamma_n \overline{\gamma_{n-1}} & -\gamma_n \kappa_{n-1} \overline{\gamma_{n-2}} & \cdots & \cdots & -\gamma_n \kappa_{n-1} \cdots \kappa_2 \overline{\gamma_1} & -\gamma_n \kappa_{n-1} \cdots \kappa_1 \overline{\gamma_0} \\ 0 & \kappa_{n-1} & -\gamma_{n-1} \overline{\gamma_{n-2}} & \cdots & \cdots & -\gamma_{n-1} \kappa_{n-2} \cdots \kappa_2 \overline{\gamma_1} & -\gamma_{n-1} \kappa_{n-2} \cdots \kappa_1 \overline{\gamma_0} \\ 0 & 0 & \kappa_{n-2} & \cdots & \cdots & -\gamma_{n-2} \kappa_{n-3} \cdots \kappa_2 \overline{\gamma_1} & -\gamma_{n-2} \kappa_{n-3} \cdots \kappa_1 \overline{\gamma_0} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & & \ddots & \kappa_2 & -\gamma_2 \overline{\gamma_1} & -\gamma_2 \kappa_1 \overline{\gamma_0} \\ 0 & 0 & \cdots & 0 & 0 & \kappa_1 & -\gamma_1 \overline{\gamma_0} \end{array} \right)$$

## Example: scalar discrete-time lossless systems (2)

- Here:  $|\gamma_0| = 1$ .  
For  $k = 1, 2, \dots, n$  we have  $\kappa_k := \sqrt{1 - |\gamma_k|^2}$ , with  $|\gamma_k| < 1$ .
- This realization matrix  $R$  can be factored as a **product of unitary Householder matrices**:

$$R = \Gamma_n \Gamma_{n-1} \cdots \Gamma_2 \Gamma_1 \Gamma_0,$$

with

$$\Gamma_0 = \begin{pmatrix} I_n & 0 \\ 0 & \overline{\gamma_0} \end{pmatrix}, \quad \Gamma_k = \begin{pmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & \overline{\gamma_k} & \kappa_k & 0 \\ 0 & \kappa_k & -\gamma_k & 0 \\ 0 & 0 & 0 & I_{k-1} \end{pmatrix}$$

- The matrix  $(B, A)$  and hence also the reachability matrix  $K$  are both **positive upper triangular**.

## Example: scalar discrete-time lossless systems (3)

- The function  $G_k(z)$  of degree  $k$  is related to the function  $G_{k-1}$  of degree  $k - 1$  by means of the (reversed) Schur recursion:

$$G_k(z) = \frac{\overline{\gamma_k}z + G_{k-1}(z)}{z + \gamma_k G_{k-1}(z)}.$$

- The parameters  $\gamma_k$  are called **Schur parameters**. They also show up as the reflection coefficients in the Levinson algorithm for the AR-system associated with the denominator polynomial of  $G(z)$ .

## PARAMETERIZATION OF DISCRETE-TIME LOSSLESS SYSTEMS WITH THE TANGENTIAL SCHUR ALGORITHM

# Linear fractional transformations

- LFTs are a well established tool in interpolation theory, occurring extensively in representation formulas for the solution of various interpolation problems, see [Dym \(1989\)](#); [Ball, Gohberg and Rodman \(1990\)](#).
- LFTs show up in various different forms in the literature. We employ the following form:

$$\mathcal{T}_\Theta : G \mapsto (\Theta_4 G + \Theta_3)(\Theta_2 G + \Theta_1)^{-1}.$$

where  $\Theta(z)$  is a  $2p \times 2p$  rational matrix, partitioned as

$$\Theta = \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix},$$

and  $G(z)$  is a  $p \times p$  rational matrix.

## Some properties of LFTs

- The **group property** for composition of LFTs:

$$\mathcal{T}_\Phi \circ \mathcal{T}_\Psi = \mathcal{T}_{\Phi\Psi}.$$

- The **inverse** of an LFT:

$$\mathcal{T}_\Theta^{-1} = \mathcal{T}_{\Theta^{-1}}$$

- **Equivalence of LFTs**: it holds that

$$\mathcal{T}_\Phi = \mathcal{T}_\Psi$$

if and only if there exists a scalar function  $\lambda(z)$  for which  $\Phi(z) = \lambda(z)\Psi(z)$ . This still holds true when the domain for  $G(z)$  is restricted to just the real lossless functions.

- An LFT with a **block-diagonal** matrix  $\Theta(z)$  corresponds to a product formula: if  $\Theta_2 = \Theta_3 = 0$  then  $\mathcal{T}_\Theta(G) = \Theta_4 G \Theta_1^{-1}$ .

## J-inner matrices

We shall be using  $2p \times 2p$  rational matrix functions  $\Theta(z)$  which are **J-inner**:

$$\Theta(z)^* J \Theta(z) \leq J, \quad \text{for } |z| < 1,$$

$$\Theta(z)^* J \Theta(z) = J, \quad \text{for } |z| = 1,$$

$$\Theta(z)^* J \Theta(z) \geq J, \quad \text{for } |z| > 1.$$

Here:  $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$ .

The **inverse** of a J-inner function  $\Theta(z)$  is given by  $\Theta(z)^{-1} = J \Theta^*(z^{-1}) J$ .

A **constant J-unitary** matrix  $\Theta$  satisfies  $\Theta^* J \Theta = J$ .

## Constant $J$ -unitary matrices

- **THEOREM** Every constant  $J$ -unitary matrix  $M$  can be represented in a unique way as

$$M = H(E) \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where  $P$  and  $Q$  are  $p \times p$  unitary matrices and  $H(E)$  denotes the **Halmos extension** of a strictly contractive  $p \times p$  matrix  $E$  (i.e., such that  $I - E^*E > 0$ ).

- This Halmos extension  $H(E)$  is defined by

$$H(E) = \begin{pmatrix} (I - EE^*)^{-1/2} & E(I - E^*E)^{-1/2} \\ E^*(I - EE^*)^{-1/2} & (I - E^*E)^{-1/2} \end{pmatrix}.$$

It holds that  $H(E)$  is Hermitian,  $J$ -unitary and invertible with inverse  $H(E)^{-1} = H(-E)$ .

# Elementary $J$ -inner factors

- **Elementary  $J$ -inner factors** are  $J$ -inner matrices of McMillan degree 1. They are characterized by the following result.

- **THEOREM** Let  $\Theta(z)$  be an elementary  $J$ -inner factor with its pole at  $z = 1/\bar{w}$ . Then  $\Theta(z)$  must be in one of the following three forms:

(1) If  $|w| \neq 1$ , then  $\Theta(z) = (I_{2p} + \left(\frac{z-w}{1-\bar{w}z} - 1\right) \frac{xx^*J}{x^*Jx})M$ ,

for some constant  $J$ -unitary  $M$  and  $x$  such that  $\frac{x^*Jx}{1-|w|^2} > 0$ .

(2) If  $|w| = 1$ , then  $\Theta(z) = (I_{2p} + \delta \frac{(z+w)}{(z-w)} xx^*J)M$ ,

for some constant  $J$ -unitary  $M$ ,  $x$  such that  $x^*Jx = 0$ , and some real  $\delta > 0$ .

(3) If  $w = \infty$ , then  $\Theta(z) = (I_{2p} + \left(\frac{1}{z} - 1\right) \frac{xx^*J}{x^*Jx})M$ ,

for some constant  $J$ -unitary  $M$  and  $x$  such that  $x^*Jx < 0$ .

# Importance of LFTs with $J$ -inner matrix functions for the construction of lossless functions

- **THEOREM** Let  $\Theta(z)$  be  $J$ -inner of McMillan degree  $m$ . If  $G(z)$  is lossless of degree  $n$ , then  $\widehat{G} = \mathcal{T}_\Theta(G)$  is well-defined, lossless and of degree  $\leq n + m$ .
- For a proof: see [Ball, Gohberg and Rodman \(1990\)](#).
- If  $\Theta(z)$  happens to be constant  $J$ -unitary, then the associated LFT  $\mathcal{T}_\Theta(G)$  does not change the degree of the lossless function  $G(z)$ .

# The tangential Schur algorithm

- The **tangential Schur algorithm** was used in [Alpay, Baratchart and Gombani \(1994\)](#) to construct an atlas of overlapping parameterizations for the class of rational inner functions. These are bijectively related to rational lossless functions.
- Is **recursive** in the degree of the lossless functions.
- The tangential Schur algorithm starts from a given lossless function of degree  $n$  and uses **interpolation conditions** to break it down, in  $n$  iterations, to a lossless function of degree 0 (i.e., to a constant unitary matrix).
- In each recursion step **an LFT is used**, involving an elementary  $J$ -inner factor, which reduces the degree exactly by one.
- The **parameterization procedure** involves the **reversed** tangential Schur algorithm, which also involves LFTs associated with elementary  $J$ -inner factors.

## The elementary $J$ -inner factors

In the framework of [HOP \(2006\)](#), **elementary  $J$ -inner factors** are used which have a pole outside the closed unit disk, at  $z = \bar{w}^{-1}$ . They are of the form:

$$\Theta_{u,v,w,\xi,H}(z) = \left( I_{2p} + \left( \frac{(z-w)(1-\bar{w}\xi)}{(1-\bar{w}z)(\xi-w)} - 1 \right) \frac{\begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}^* J}{(1-\|v\|^2)} \right) H,$$

where

- (1)  $w$  is a scalar interpolation point with  $|w| < 1$ ,
- (2)  $u$  is a direction vector, normalized by  $\|u\| = 1$ ,
- (3)  $v$  is the **Schur parameter vector**, satisfying  $\|v\| < 1$ ,
- (4)  $\xi$  is a normalization point for  $\Theta(z)$  with  $|\xi| = 1$ ,
- (5)  $H$  is a constant  $J$ -unitary matrix; note that  $H = \Theta(\xi)$ .

## Structure of a recursion step

- **THEOREM** The  $J$ -inner matrix function  $\Theta_{u,v,w,\xi,H}(z)$  can be factored as:

$$\Theta_{u,v,w,\xi,H}(z) = H(uv^*)S_{u,w}(z)S_{u,w}(\xi)^{-1}H(uv^*)^{-1}H,$$

where  $H(uv^*)$  denotes the Halmos extension of the strictly contractive matrix  $uv^*$ , and where

$$S_{u,w}(z) := \begin{pmatrix} I_p + \left( \frac{z-w}{1-\bar{w}z} - 1 \right) uu^* & 0 \\ 0 & I_p \end{pmatrix}.$$

- The LFT is composed of the LFTs for each of the individual factors.
- Only the LFT involving the factor  $S_{u,w}(z)$  changes the degree. As  $S_{u,w}(z)$  is block-diagonal, the LFT for this factor performs **postmultiplication by a lossless matrix of degree 1**: it takes a lossless function  $G(z)$  to  $G(z)(I_p + \left( \frac{1-\bar{w}z}{z-w} - 1 \right) uu^*)$ .

## Structure of a recursion step (2)

- **THEOREM** Let  $G(z)$  be lossless of degree  $n$ . Then

$$\widehat{G}(z) = \mathcal{T}_{\Theta_{u,v,w,\xi,H}(z)}(G(z))$$

is lossless of  $n + 1$ , satisfying the interpolation condition

$$\widehat{G}(\overline{w}^{-1})u = v,$$

- Conversely, let  $\widehat{G}(z)$  be lossless of degree  $n + 1$ , such that it satisfies an interpolation condition of the form  $\widehat{G}(\overline{w}^{-1})u = v$  for some  $w, u$  and  $v$  with  $|w| < 1$ ,  $\|u\| = 1$  and  $\|v\| < 1$ . Let  $|\xi| = 1$  and let  $H$  be a constant  $J$ -unitary matrix. Then  $\widehat{G}(z)$  admits the representation

$$\widehat{G} = \mathcal{T}_{\Theta_{u,v,w,\xi,H}}(G),$$

for a unique lossless function  $G(z)$  of degree  $n$ .

## Special cases

- If one chooses  $w = 0$ , then  $\widehat{G}(\infty) = \widehat{D}$  and the interpolation condition takes the form  $\widehat{D}u = v$ . This case is often encountered when constructing canonical forms with additional triangularity properties.
- If one chooses  $w$  to be a pole of  $\widehat{G}(z)$ , then  $\widehat{G}(\overline{w}^{-1})$  is singular. If  $u$  is chosen in the kernel, one finds  $v = 0$ . Then with  $H = S_{u,w}(\xi)$  one obtains:

$$\widehat{G}(z) = G(z) \left( I_p + \left( \frac{1 - \overline{w}z}{z - w} - 1 \right) uu^* \right).$$

When repeated, this provides a **Potapov decomposition** of  $\widehat{G}(z)$

# Tangential Schur algorithm

**THEOREM** Let  $G^{(n)}$  be lossless of degree  $n$ .

For  $k = n, \dots, 2, 1$ , choose:

- interpolation points  $|w_k| < 1$ ,
- constants  $|\xi_k| = 1$ ,
- mappings  $H_k : (u, v, w, \xi) \mapsto H_k(u, v, w, \xi)$  with  $H_k(u, v, w, \xi)$  constant  $J$ -unitary for  $\|u\| = 1$ ,  $\|v\| < 1$ ,  $|w| < 1$  and  $|\xi| = 1$ .

Then for  $k = n, \dots, 2, 1$ , there exist vectors  $u_k$  with  $\|u_k\| = 1$ , such that the vectors  $v_k$  which are constructed recursively by the following formulas, all have  $\|v_k\| < 1$ :

$$v_k := G^{(k)}(\overline{w_k}^{-1})u_k,$$

$$\Theta_k := \Theta(u_k, v_k, w_k, \xi_k, H_k(u_k, v_k, w_k, \xi_k)),$$

$$G^{(k-1)} := \mathcal{T}_{\Theta_k}^{-1}(G^{(k)}).$$

With such a choice of the unit vectors  $u_k$  ( $k = n, \dots, 2, 1$ ) each of the functions  $G^{(k)}$  is lossless of degree  $k$  and one can write

$$G^{(n)} = \mathcal{T}_{\Theta_n}(\mathcal{T}_{\Theta_{n-1}}(\dots(\mathcal{T}_{\Theta_1}(G^{(0)}))\dots)) = \mathcal{T}_{\Theta_n \Theta_{n-1} \dots \Theta_1}(G^{(0)}),$$

where  $G^{(0)}$  is a constant unitary matrix.

# Parameterization of lossless systems with the reversed tangential Schur algorithm

- The **reversed tangential Schur algorithm** is used to generate a parameterized chart of lossless systems.
- The Schur vectors  $v_1, v_2, \dots, v_n$  and the unitary matrix  $G^{(0)}$  serve as the **parameters**.
- The set of values for  $w_k, u_k, \xi_k$  and the set of mappings  $H_k$  at the  $n$  recursion steps serve to **index** a generic chart for the differentiable manifold of lossless functions of degree  $n$ , provided that the mappings  $H_k$  are sufficiently smooth.
- Such a chart involves  $np + \frac{1}{2}p(p-1)$  real degrees of freedom in the real case, and  $2np + p^2$  real degrees of freedom in the complex case.

## State-space implementation with unitary matrix multiplication

- Recall that in the **scalar discrete-time** lossless case, a parameterization was found with Schur parameters, for which the realization matrix is composed of a product of elementary unitary matrices:

$$R = \Gamma_n \Gamma_{n-1} \cdots \Gamma_2 \Gamma_1 \Gamma_0.$$

- We now **generalize** this to the **MIMO discrete-time** lossless case.
- We consider what happens when we perform the following action:

$$\begin{pmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & I_n \end{pmatrix},$$

starting from a balanced realization  $(A, B, C, D)$  of a lossless  $p \times p$  function  $G(z)$  of order  $n$ , and **unitary**  $(p+1) \times (p+1)$  matrices  $U$  and  $V$ .

## State-space implementation with unitary matrix multiplication (2)

**THEOREM** Consider the transfer function  $\tilde{G}(z)$  associated with the realization  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  obtained by the previous action. It holds that:

- (1)  $\tilde{G}(z)$  is lossless of degree  $\leq n + 1$ .
- (2) If minimality happens to hold, then the realization for  $\tilde{G}(z)$  is balanced.
- (3)  $\tilde{G}(z)$  does not depend on the choice of balanced realization  $(A, B, C, D)$  but only on the lossless function  $G(z)$ . It is given by:

$$\tilde{G}(z) = F_1(z) + \frac{F_2(z)F_3(z)}{z - F_4(z)},$$

where

$$\begin{pmatrix} F_1(z) & F_2(z) \\ F_3(z) & F_4(z) \end{pmatrix} = V \begin{pmatrix} 1 & 0 \\ 0 & G(z) \end{pmatrix} U^*.$$

with  $F_1(z)$  of size  $p \times p$ , and so on.

We introduce the mapping  $\mathcal{F}_{U,V} : G(z) \mapsto \tilde{G}(z)$  for convenience.

## Tangential Schur recursion in state-space realization terms

**THEOREM** An LFT  $\mathcal{T}_{\Theta_{u,v,w,\xi,H}(z)}$  coincides with a mapping  $\mathcal{F}_{U,V}$  with unitary  $U$  and  $V$ , if and only if

$$\Theta_{u,v,w,\xi,H}(z) = H(uv^*)S_{u,w}(z)H(\bar{w}uv^*) \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

for some  $\|u\| = 1$ ,  $\|v\| < 1$ ,  $|w| < 1$ , and some  $p \times p$  unitary matrices  $P$  and  $Q$ .

Equivalently:

$$H = H(uv^*)S_{u,w}(\xi)H(\bar{w}uv^*) \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}.$$

## Tangential Schur recursion in state-space realization terms

In that case one can take  $U = \hat{U} \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$  and  $V = \hat{V} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ , where

$$\hat{U} = \begin{pmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}} u & I_p - \left(1 + \frac{w\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}}\right) uu^* \\ \frac{\bar{w}\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}} u^* \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}} v & I_p - \left(1 - \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}}\right) \frac{vv^*}{\|v\|^2} \\ \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & -\frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}} v^* \end{pmatrix}$$

## SUMMARY

We have constructed balanced state-space realizations for discrete-time lossless systems:

- **Parameterized** by the Schur parameter vectors  $v_k$ .
- There are many generic **charts**, indexed by  $w_k$ ,  $u_k$ ,  $\xi_k$  and  $H_k$ , which can be freely chosen (only subject to the standard restrictions).
- This construction is **recursive** on the degree.
- One can always choose  $H_k$  to let the LFTs be mappings  $\mathcal{F}_{U,V}$ . Then the balanced realization matrix is obtained as a **product of unitary matrices**.
- Choices can always be made to obtain **triangular forms** (staircase structures) in  $(B, A)$  and in  $K$ , which behave well under **truncation**.

## Some special cases

- For  $w = 0$  it follows that

$$\hat{U} = \begin{pmatrix} u & I_p - uu^* \\ 0 & u^* \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} v & I_p - (1 - \sqrt{1 - \|v\|^2}) \frac{vv^*}{\|v\|^2} \\ \sqrt{1 - \|v\|^2} & -v^* \end{pmatrix}.$$

With  $u = 1$  in the SISO case, the scalar discrete-time balanced canonical form is recovered.

When  $u$  is chosen from the standard basis vectors  $\{e_1, e_2, \dots, e_p\}$ , this produces “subdiagonal pivot structures”. Special choices yield staircase structures (generalizing positive upper triangularity) for  $(B, A)$  and for  $K = (B, AB, \dots, A^{n-1}B)$ .

- When reversing the procedure, choosing  $w$  as a pole of  $G(z)$ ,  $u$  in the kernel of  $G(\bar{w}^{-1})$  and  $v = 0$ , yields:

$$\hat{U} = \begin{pmatrix} \sqrt{1 - |w|^2}u & I_p - (1 + w)uu^* \\ \bar{w} & \sqrt{1 - |w|^2}u^* \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} 0 & I_p \\ 1 & 0 \end{pmatrix}.$$

This is useful to produce a state-space canonical form with  $A$  in triangular Schur form, and to switch between parameter charts (see [Olivi and Marmorat \(2003\)](#)) in case of numerical ill-conditioning.

## Some further comments

- The **real case** is covered conveniently by requiring  $w$ ,  $u$ ,  $v$ ,  $\xi$  and  $H$  to be real.
- The **boundary of a chart** contains lower order systems. It also contains systems of the **same** degree.
- Recursion steps can be taken which change the degree by more than 1 (**Nudelman interpolation**).
- A balanced realization for  $\hat{G} = \mathcal{T}_{\Theta_{u,v,w,\xi,H}}(G)$  can also be obtained by applying an LFT directly to an extended unitary realization matrix  $R$  for  $G$ , using a “ $J$ -balanced” realization for  $\Theta_{u,v,w,\xi,H}(z^{-1})$  to define the LFT. This produces different balanced realizations.
- The charts in the current framework involve interpolation points that are not on the stability boundary. Triangular structures are obtained for  $w = 0$  in the discrete-time case. In the continuous-time case, the triangular structures of Ober’s canonical form correspond to interpolation on the stability boundary, at infinity. This can be generalized to the MIMO continuous-time case.

## APPLICATIONS

## APPLICATION: Parameterization of stable systems

**Algorithm** to bring a minimal and stable system  $(A, B, C, D)$  into canonical form (compare with the algorithm for a balanced canonical form):

- 1 Compute the controllability Gramian  $W_c$ .
- 2 Factor  $W_c$  as  $W_c = P^*P$  (e.g., SVD or Cholesky decomposition).
- 3 Use  $P$  to transform  $(A, B, C, D)$  to get  $W_c = I_n$  (**input normality**).
- 4 Compute the reachability matrix  $K$ .
- 5 Perform  $QR$ -decomposition on  $K$ .
- 6 Use  $Q$  to transform  $(A, B, C, D)$ , to make  $W_c = I_n$  and  $K$  positive upper triangular.

## Properties of the canonical form

Some **properties** of the resulting canonical form:

- A parameterization is obtained by:
  - (1) parameterizing the input normal pairs  $(A, B)$  which have a positive upper triangular reachability matrix;
  - (2) taking all the entries of  $C$  and  $D$  as free parameters.
- For (1) a parameterization follows directly from the parameterization of lossless systems, by taking  $G^{(0)}$  to be the identity matrix  $I_p$ .
- State vector **truncation** yields an approximation which is again in canonical form.
- Lower order systems occur if and only if observability is lost.
- In the MIMO case, Kronecker structures can also be handled in this way.
- Useful for system identification (separable least squares) and  $L_2$ -approximation.

## APPLICATION: Separable least squares in system identification

- **Separable least squares** (SLS): first introduced in [Golub and Pereyra \(1973\)](#); see also [Bruls, Chou, Haverkamp and Verhaegen \(1999\)](#).
- The optimization/estimation/identification problem: **nonlinear least squares**.
- Fixing  $A$  and  $C$ , the problem becomes **linear least squares** for  $B$  and  $D$ .
- $B$  and  $D$  are optimized in terms of  $A$  and  $C$  and then eliminated by substitution.
- New optimization problem: over **output normal** pairs  $(C, A)$ .
- Output normal pairs can be conveniently parameterized with Schur vectors, using products of unitary matrices.

## APPLICATION: $L_2$ -approximation

- In [Douglas, Shapiro and Shields \(1970\)](#), a strictly proper stable transfer function  $H(z)$  is factored into a product  $H(z) = G(z)P(z)$  with  $G(z)$  lossless and  $P(z)$  unstable.
- For fixed  $G(z)$ , the  $L_2$ -approximation problem can be solved for  $P(z)$ . Then  $P(z)$  is eliminated by substitution.
- This leaves an optimization problem: over lossless functions  $G(z)$ .
- An equivalent procedure can be carried out in state-space terms: for fixed  $(A, B)$  one can optimize for  $C$  and  $D$ , then leaving an optimization problem over input normal pairs  $(A, B)$ .

## $L_2$ -approximation (2)

- Transformation between continuous-time and discrete-time:

Let  $\widehat{G}(s) = \widehat{C}(sI_n - \widehat{A})^{-1}\widehat{B}$  be given, and define

$$G(z) = \frac{\sqrt{2}}{z-1} \widehat{G}\left(\frac{z+1}{z-1}\right).$$

Then a realization  $G(z) = C(zI_n - A)^{-1}B$  is obtained by:

$$\begin{aligned} A &= (\widehat{A} + I_n)(\widehat{A} - I_n)^{-1}, \\ B &= \widehat{B}, \\ C &= -\sqrt{2}\widehat{C}(\widehat{A} - I_n)^{-1}. \end{aligned}$$

- $\widehat{G}(s)$  and  $G(z)$  have the same degree.
- The (discrete-time)  $L_2$ -norm of  $G(z)$  equals the (continuous-time)  $L_2$ -norm of  $\widehat{G}(s)$ .
- The inverse transformation is given by:

$$\widehat{G}(s) = \frac{\sqrt{2}}{s-1} G\left(\frac{s+1}{s-1}\right).$$

# RARL2

The **RARL2** software:

- **Software** developed at INRIA for  $L_2$ -approximation. Available upon request.
- Takes a (high order) system or frequency data as input.
- Is implemented with unitary product forms.
- Has a chart selection strategy (monitoring the norms of the Schur parameter vectors  $v_k$ ).
- Can proceed recursively with respect to an increasing degree.
- Has good numerical performance, due to unitary matrix operations.

## APPLICATION: Orthogonal wavelets from filter banks

- See [Strang and Nguyen \(1996\)](#).
- In this setting, the computation of **approximation coefficients** and **wavelet coefficients** can be performed by: linear **filtering** of the given signal, followed by **down-sampling**.
- Then the two filters (low-pass and high-pass) must be **power complementary** to get **orthogonal** wavelet and scaling function bases.
- **Equivalently**, one can also proceed by: splitting the signal into two phases (down-sampling), followed by filtering with a  $2 \times 2$  **polyphase filter**.
- Orthogonality now requires this polyphase filter to be **lossless**.
- Additional properties: FIR and vanishing moments.

**Thank you!**

**Questions?**